

Secure Two-Party Sampling Primitives are either Useless or Complete

Ye Wang
Mitsubishi Electric Research Laboratories
Cambridge, MA, USA
Email: yewang@merl.com

Prakash Ishwar
Boston University
Boston, MA, USA
Email: pi@bu.edu

Shantanu Rane
Mitsubishi Electric Research Laboratories
Cambridge, MA, USA
Email: rane@merl.com

Abstract—In the secure two-party sampling problem, two parties wish to generate outputs with a desired joint distribution via an interactive protocol, while ensuring that neither party learns more than what can be inferred from only their own output. For semi-honest parties and information-theoretic privacy guarantees, it is well-known that if only noiseless communication is available, then only the “trivial” joint distributions, for which common information equals mutual information, can be securely sampled. We consider the problem where the parties may also interact via a given set of general communication primitives (multi-input/output channels). Our feasibility characterization of this problem can be stated as a zero-one law: primitives are either complete (enabling the secure sampling of any distribution) or useless (only enabling the secure sampling of trivial distributions). Our characterization of the complete primitives also extends to the more general class of secure two-party computation problems.

I. INTRODUCTION

We consider the problem of secure two-party sampling, an important subclass of secure computation problems where two parties named Alice and Bob wish to securely generate outputs according to a desired joint distribution. Security means that Alice and Bob should produce outputs with the correct distribution while also ensuring privacy, in the sense that neither party learns anything about the other party’s output besides what can be inferred through any inherent correlation. We restrict our attention to “semi-honest” parties who will faithfully execute a given protocol, but attempt to extract additional information from their views of the execution. However, we require information-theoretic privacy, providing unconditional security guarantees against adversaries with even unbounded computational power. The aim is to construct interactive protocols that allow the parties to produce outputs while ensuring the above security goals. In these protocols, the parties are allowed unlimited interaction via noise-free communication as well as via a given set of communication primitives, which are general memoryless two-way channels where each party may have an input and an output.

Our main result is the general feasibility characterization of this problem, that is, determining which distributions can be securely sampled via a protocol using any specified set of primitives. Interestingly, this problem exhibits a zero-one law, in the sense that any set of primitives is either “complete” (allowing any distribution to be securely sampled) or “useless” (allowing only a set of “trivial” distributions to be sampled).

The trivial distributions are those that can be securely sampled “from scratch”, that is, when only noise-free communication is available and no other primitives can be used. This set of joint distributions was characterized in [1] as those for which the mutual information is equal to the common information¹. The class of complete primitives in this secure sampling problem are also the primitives that are “complete” in the more general secure two-party *computation* problem (allowing any two-party computation to be securely performed).

In the larger class of secure two-party *computation* problems, the parties initially have inputs from which they wish to securely compute outputs according to a (possibly randomized) function. Secure sampling is the subclass of problems where the inputs are null (or constant), and the desired function produces random outputs. It is well-known that not all functions can be securely computed by two-parties from scratch (see [4]). Thus, much work has been done on identifying the complete primitives that enable any secure computation, and on general protocol constructions using those primitives. Two key results are useful for general protocol construction: oblivious transfer² is a complete primitive [5], and secure computation of oblivious transfer can be leveraged to perform general secure computation [6]. Hence, a significant focus in the literature has been the identification of which primitives within certain subclasses enable oblivious transfer and hence are complete (along with proposing efficient constructions): one-way channels (primitives with one input and one output) [7], [8], [9], joint sources (primitives with no inputs) [9], and primitives with only one output or a common output [10]. Our characterization of the complete primitives subsumes these specialized characterizations.

II. PROBLEM FORMULATION

A. Secure Two-Party Sampling Protocols

Alice and Bob wish to securely generate samples from the distribution $P_{X,Y}$ over the finite alphabet $\mathcal{X} \times \mathcal{Y}$. To realize this goal, they execute a two-party sampling protocol at the end of which Alice outputs $\hat{X} \in \mathcal{X}$ and Bob outputs $\hat{Y} \in \mathcal{Y}$.

¹This property is equivalent for the Wyner [2] and Gács-Körner [3] notions of common information.

²Oblivious transfer is the primitive or function where Alice’s has a two-bit input and no output, and Bob’s output is a bit selected from Alice’s input by his binary input.

A protocol may involve multiple rounds of local computation interspersed with rounds of interaction via error-free communication or through one of the available communication primitives. A *communication primitive* is a channel with input (A, B) in the finite alphabet $\mathcal{A} \times \mathcal{B}$, output (U, V) in the finite alphabet $\mathcal{U} \times \mathcal{V}$, and a conditional distribution $P_{U,V|A,B}$. Each primitive usage is “memoryless” and Alice controls input A and receives output U while Bob controls input B and receives output V . After the protocol terminates, Alice and Bob generate their respective outputs via deterministic functions of their respective *views* of the protocol. A party’s view consists of its local computations, messages sent and received, and inputs to and outputs from the chosen primitives.

For simplicity, we only consider protocols that terminate in a fixed (deterministic) number of rounds n , but do not put a bound on n . A protocol consists of a sequence of steps that governs how the views of the parties can evolve during the protocol’s execution. The initial views of Alice and Bob are constant and respectively denoted by $R_0 = S_0 = 0$. Let $(R_1, S_1), \dots, (R_n, S_n)$ denote the sequence of evolution of their views over n rounds. In each round t of the protocol, the evolution of views from (R_{t-1}, S_{t-1}) to (R_t, S_t) occurs via one of three possible structured mechanisms: local computation, error-free message passing, or usage of a primitive (if available).

- (Local computation) $R_t = (R_{t-1}, A)$ and $S_t = (S_{t-1}, B)$, where $A \leftrightarrow R_{t-1} \leftrightarrow S_{t-1} \leftrightarrow B$ is a Markov chain.
- (Message passing) $R_t = (R_{t-1}, g(S_{t-1}))$ and $S_t = (S_{t-1}, f(R_{t-1}))$, where f and g are deterministic functions.
- (Primitive usage) $R_t = (R_{t-1}, U)$ and $S_t = (S_{t-1}, V)$, where (U, V) are the outputs of one of the given communication primitives, with inputs $A = f(R_{t-1})$ and $B = g(R_{t-1})$ generated via deterministic functions f and g , and $P_{U,V|A,B}$ corresponds to the distribution governing the primitive used. The memoryless behavior of the primitives implies that $(U, V) \leftrightarrow (A, B) \leftrightarrow (R_{t-1}, S_{t-1})$ is a Markov chain.

After n rounds, outputs are generated deterministically from the final views, that is, $\hat{X} = \phi(R_n)$ and $\hat{Y} = \psi(S_n)$, for some functions ϕ and ψ .

B. Security Definitions

A protocol is called ϵ -correct if the variational distance between the distribution of the output and the desired distribution does not exceed ϵ :

$$d(P_{\hat{X}, \hat{Y}}, P_{X, Y}) := \frac{1}{2} \sum_{(x, y) \in \mathcal{X} \times \mathcal{Y}} |P_{\hat{X}, \hat{Y}}(x, y) - P_{X, Y}(x, y)| \leq \epsilon.$$

A protocol is δ -private if and only if the information leakage of the final views satisfies

$$I(R_n; \hat{Y} | \hat{X}) + I(S_n; \hat{X} | \hat{Y}) \leq \delta.$$

We will say that a protocol is (ϵ, δ) -secure if it is ϵ -correct and δ -private. A distribution $P_{X, Y}$ can be *securely sampled*

using a given set of primitives if and only if for any $\epsilon, \delta > 0$ there exists an (ϵ, δ) -secure protocol for sampling $P_{X, Y}$ using the specified primitives.

The distributions that can be securely sampled via protocols that use only error-free communication (and no other communication primitives) will be called *trivial*, since they can always be securely sampled regardless of the other primitives available.

A primitive is said to be *complete* if any distribution can be securely sampled using that primitive. A *set* of primitives is said to be *useless* if only the trivial distributions can be securely sampled using that set of primitives.

C. Secure Two-Party Computation

The more general secure two-party *computation* problem can be formulated similarly as above but with the following additional generalizations. Alice and Bob respectively start with inputs Q and T with joint distribution $P_{Q, T}$ over the finite alphabet $\mathcal{Q} \times \mathcal{T}$. The objective of a protocol is to produce an output (\hat{X}, \hat{Y}) that securely simulates the (in general randomized) function $P_{X, Y|Q, T}$. The initial views of the parties are $(R_0, S_0) := (Q, T)$. The conditions for ϵ -correctness and δ -privacy are modified to $d(P_{\hat{X}, \hat{Y}|Q, T}, P_{X, Y|Q, T}) \leq \epsilon$ and $I(R_n; \hat{Y}, T | \hat{X}, Q) + I(S_n; \hat{X}, Q | \hat{Y}, T) \leq \delta$ respectively. A function $P_{X, Y|Q, T}$ is said to be *securely computable* if for all $\epsilon, \delta > 0$, there exists a protocol that is (ϵ, δ) -secure for all input distributions $P_{Q, T}$. A primitive that allows any function to be securely computed (simulated) is called *complete* (for secure computation).

Observation: Since secure sampling is a special case of secure computation, a primitive that is complete for secure computation is also complete for secure sampling. Interestingly, the reverse is also true (see the achievability sketch for Theorem 1 in Section V-B). Thus, a primitive is complete for secure computation (enabling the secure computation of any function) if and only if it is complete for secure sampling (enabling the secure sampling of any distribution). Hence, we can call a primitive complete without specifying whether it is for secure sampling or computation.

III. CHARACTERIZATION RESULTS

A. Preliminaries

Common information plays a key role in the characterizations of both the secure sampling and computation problems. There are two related (and somewhat complementary) notions of common information, one introduced by Wyner [2] and the other introduced by Gács-Körner [3]. We will review only the Wyner common information here to allow us to quickly state our results, and leave Gács-Körner common information and other related concepts to be reviewed later in Section IV.

The Wyner common information of two random variables (X, Y) is given by

$$C(X; Y) := \min_{Z: I(X; Y|Z)=0} I(X, Y; Z),$$

where the minimum can be attained by a $Z \in \mathcal{Z}$ with $|\mathcal{Z}| \leq |\mathcal{X} \times \mathcal{Y}|$ [2]. This quantity characterizes the solution of

the Gray-Wyner source coding problem. Note that, in general, $C(X; Y) \geq I(X; Y)$ [2].

It follows from the results of [1] and the continuity of Wyner common information (see Lemma 4 in Section IV), that the trivial distributions, i.e., those which can be securely sampled from scratch, are precisely those where $C(X; Y) = I(X; Y)$ (see Lemma 1 in Section IV for equivalent conditions). We will hence use the terms *trivial* (and *non-trivial*) to refer to joint distributions $P_{X,Y}$ which do (and, respectively, do not) satisfy $C(X; Y) = I(X; Y)$.

B. Main Results

Our main result characterizes the complete and useless sets of primitives and establishes the following zero-one law: any set of primitives is either complete or useless.

Theorem 1. *A primitive $P_{U,V|A,B}$ is complete if and only if there exist random variables (R, S) and functions $f : \mathcal{R} \rightarrow \mathcal{A}$, $g : \mathcal{S} \rightarrow \mathcal{B}$ such that $C(R; S) = I(R; S)$ and $C(R, U; S, V) > I(R, U; S, V)$, where $A = f(R)$ and $B = g(S)$. Further, any set of incomplete primitives is useless.*

An interpretation of a complete primitive is that its usage can produce a non-trivial distribution on the resultant views $((R, U), (S, V))$ while starting from prior views (R, S) that have a trivial distribution.

The following corollary of our main result characterizes which distributions can be securely sampled using a given set of primitives.

Corollary 1. *Given any set of primitives, if at least one is complete (see conditions in Theorem 1), then any distribution $P_{X,Y}$ can be securely sampled. Otherwise, only the trivial distributions, where $C(X; Y) = I(X; Y)$, can be securely sampled.*

IV. PROPERTIES OF COMMON INFORMATION

This section reviews key concepts and results needed to establish our main results. They are, however, also of independent interest.

The graphical representation of $P_{X,Y}$ is the bipartite graph with an edge between $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ if and only if $P_{X,Y}(x, y) > 0$. The *common part* of two random variables (X, Y) , denoted by $W_{X,Y}$, is the (unique) label of the connected component of the graphical representation of $P_{X,Y}$ in which (X, Y) falls. Note that $W_{X,Y}$ is a deterministic function of X alone and also a deterministic function of Y alone.

The Gács-Körner common information of two random variables (X, Y) is given by $K(X; Y) := H(W_{X,Y})$ [3]. The operational significance of $K(X; Y)$ is that it is the maximum number of common bits per symbol that can be independently extracted from X and Y . Note that, in general, $K(X; Y) \leq I(X; Y)$ [3].

While it may be tedious, in general, to solve the optimization problem that defines Wyner common information, one can conveniently check if it is equal to its lower bound by using its well-known relationship to Gács-Körner common information and other properties given in the following lemma (see [11]).

Lemma 1. *For any random variables (X, Y) , the following are equivalent:*

- 1) $C(X; Y) = I(X; Y)$,
- 2) $K(X; Y) = I(X; Y)$,
- 3) *There exists Z such that $Z \leftrightarrow X \leftrightarrow Y$, $Z \leftrightarrow Y \leftrightarrow X$, and $X \leftrightarrow Z \leftrightarrow Y$ are all Markov chains,*
- 4) $X \leftrightarrow W_{X,Y} \leftrightarrow Y$ is a Markov chain, where $W_{X,Y}$ is the common part of (X, Y) .

One can also determine whether common information is equal to mutual information by checking if conditional entropy is positive after “removing redundancies” from the random variables. To *remove redundancy* from X with respect to $P_{X,Y}$, first partition the support of P_X into equivalence classes using $P_{Y|X=x} = P_{Y|X=x'}$ as the equivalence rule for $x, x' \in \mathcal{X}$, then uniquely label these classes and define \tilde{X} as the label of the class in which X falls. Similarly, \tilde{Y} can be defined as Y with redundancies removed. Note that, by construction, $X \leftrightarrow \tilde{X} \leftrightarrow \tilde{Y} \leftrightarrow Y$ is a Markov chain.

Lemma 2. *For any random variables (X, Y) , the following are equivalent:*

- 1) $C(X; Y) = I(X; Y) = K(X; Y)$,
- 2) $H(\tilde{X}|\tilde{Y}) = 0$,
- 3) $H(\tilde{Y}|\tilde{X}) = 0$,

where (\tilde{X}, \tilde{Y}) are (X, Y) with redundancies removed.

Proof: Any $x, x' \in \mathcal{X}$ with $P_{Y|X=x} = P_{Y|X=x'}$ are clearly in the same connected component of the graphical representation of $P_{X,Y}$. If $X \leftrightarrow W_{X,Y} \leftrightarrow Y$ is a Markov chain, then for any symbols $x, x' \in \mathcal{X}$ attached to the same connected component, $P_{Y|X=x} = P_{Y|X=x'}$. Thus, given condition 1, we find that $W_{X,Y}$, \tilde{X} , and \tilde{Y} (via similar arguments) are equivalent, that is, $W_{X,Y} = f(\tilde{X}) = g(\tilde{Y})$ for some bijective functions f and g . Hence, it follows that condition 1 implies condition 2 and 3. Given condition 2, \tilde{X} is a function of \tilde{Y} , and hence a function of Y . By construction, \tilde{X} is a function of X , and $X \leftrightarrow \tilde{X} \leftrightarrow \tilde{Y} \leftrightarrow Y$ is a Markov chain. Hence, $X \leftrightarrow \tilde{X} \leftrightarrow Y$, $\tilde{X} \leftrightarrow X \leftrightarrow Y$, and $\tilde{X} \leftrightarrow Y \leftrightarrow X$ are all Markov chains and condition 1 holds by Lemma 1. Similarly, condition 3 also implies condition 1. ■

Another useful property for checking whether the Wyner common information is close to the mutual information is given in the next lemma (see [1]).

Lemma 3. *For any random variables (X, Y) , $C(X; Y) - I(X; Y) \leq \delta$ if and only if there exist Z such that $X \leftrightarrow Z \leftrightarrow Y$ is a Markov chain, and $I(Z; X|Y) + I(Z; Y|X) \leq \delta$.*

Wyner common information is a uniformly continuous functional of $P_{X,Y}$ for all $P_{X,Y}$ as established in the next lemma. The Gács-Körner common information, in contrast, is discontinuous.

Lemma 4. *If $P_{X,Y}, P_{\tilde{X},\tilde{Y}}$ are joint distributions over the same finite alphabet $\mathcal{X} \times \mathcal{Y}$ with variational distance $d(P_{\tilde{X},\tilde{Y}}, P_{X,Y}) \leq \epsilon$, then $|C(X; Y) - C(\tilde{X}; \tilde{Y})| \leq \alpha(\epsilon)$, for some function α where $\alpha(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.*

Proof: One can construct random variables $(X, Y) \sim P_{X,Y}$ and $(\hat{X}, \hat{Y}) \sim P_{\hat{X},\hat{Y}}$ such that $\Pr((\hat{X}, \hat{Y}) \neq (X, Y)) = d(P_{\hat{X},\hat{Y}}, P_{X,Y})$ [12]. Let Z be the random variable such that $C(\hat{X}; \hat{Y}) = I(X, Y; Z)$ and $X \leftrightarrow Z \leftrightarrow Y$ is a Markov chain. Next, let

$$\hat{Z} := \begin{cases} (Z, \perp, \perp), & \text{when } (\hat{X}, \hat{Y}) = (X, Y), \\ (\perp, \hat{X}, \hat{Y}), & \text{when } (\hat{X}, \hat{Y}) \neq (X, Y), \end{cases}$$

where \perp is a constant symbol not in the alphabets \mathcal{X} , \mathcal{Y} , or \mathcal{Z} . By construction, $\hat{X} \leftrightarrow \hat{Z} \leftrightarrow \hat{Y}$ is a Markov chain, and $\Pr((\hat{X}, \hat{Y}, \hat{Z}) \neq (X, Y, (Z, \perp, \perp))) \leq \epsilon$. Thus,

$$\begin{aligned} C(\hat{X}; \hat{Y}) &\leq I(\hat{X}, \hat{Y}; \hat{Z}) \\ &\leq I(X, Y; Z) + \alpha(\epsilon) \\ &= C(X; Y) + \alpha(\epsilon) \end{aligned}$$

for some $\alpha(\epsilon)$ with $\alpha(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$, where the second inequality follows due to the uniform continuity of entropy [12]. Symmetrically, we can argue that $C(\hat{X}; \hat{Y}) \leq C(\hat{X}; \hat{Y}) + \alpha(\epsilon)$, and hence $|C(X; Y) - C(\hat{X}; \hat{Y})| \leq \alpha(\epsilon)$. ■

V. PROOF OF THEOREM 1

A. Converse Result

We will show that, given any set of primitives that individually fail to satisfy the completeness conditions, only trivial distributions can be securely sampled, and hence the primitives are incomplete and useless. The first part of our converse proof is closely related to the method of monotones – functionals that are monotonic over the sequence of views – introduced in [13]. Specifically, we will show that the distributions of the views P_{R_t, S_t} will remain trivial throughout the execution of the protocol. Then, we will argue that given final views (R_n, S_n) with a trivial distribution, only “almost trivial” (in the sense of Wyner common information being close to mutual information) outputs can be securely produced by a δ -private protocol. This result, in conjunction with the continuity of Wyner common information (Lemma 4), implies that only trivial distributions can be securely sampled.

The next two lemmas establish that if we start with views (R_{t-1}, S_{t-1}) that have a trivial distribution, then the views (R_t, S_t) , after respectively local computation and message passing, must also have a trivial distribution.

Lemma 5. *Let $C(R; S) = I(R; S)$. If $A \leftrightarrow R \leftrightarrow S \leftrightarrow B$ is a Markov chain then $C(A, R; B, S) = I(A, R; B, S)$.*

Proof: Let $W_{R,S}$ be the common part of random variables (R, S) . Since $C(R; S) = I(R; S)$, it follows that $R \leftrightarrow W_{R,S} \leftrightarrow S$ is a Markov chain. Since $W_{R,S}$ is a function of R alone and S alone, it also follows that $(A, R) \leftrightarrow W_{R,S} \leftrightarrow (B, S)$ is a Markov chain. Hence $C(A, R; B, S) = I(A, R; B, S)$ by Lemma 1. ■

Lemma 6. *Let $C(R; S) = I(R; S)$. If f, g are deterministic functions then $C(R, g(S); S, f(R)) = I(R, g(S); S, f(R))$.*

Proof: Let $W_{R,S}$ be the common part of (R, S) and $Z := (W_{R,S}, f(R), g(S))$. Since Z is a function of $(R, g(S))$

alone and $(S, f(R))$ alone, $(R, g(S)) \leftrightarrow (S, f(R)) \leftrightarrow Z$ and $(S, f(R)) \leftrightarrow (R, g(S)) \leftrightarrow Z$ are both Markov chains. Since

$$\begin{aligned} I(R, g(S); S, f(R)|Z) &= I(R; S|W_{R,S}, f(R), g(S)) \\ &\leq I(R, f(R); S, g(S)|W_{R,S}) = I(R; S|W_{R,S}) = 0, \end{aligned}$$

it follows that $(R, g(S)) \leftrightarrow Z \leftrightarrow (S, f(R))$ is a Markov chain. Hence, $C(R, g(S); S, f(R)) = I(R, g(S); S, f(R))$ by Lemma 1. ■

By definition, if a primitive does not meet the completeness conditions, then for all prior views (R_{t-1}, S_{t-1}) with a trivial distribution and functions (f, g) generating primitive inputs $A = f(R_{t-1})$ and $B = g(S_{t-1})$, the resultant views $(R_t, S_t) := ((R_{t-1}, U), (S_{t-1}, V))$ after using the primitive also have a trivial distribution. Combining this observation with Lemmas 5 and 6, and noting that the initial views (R_0, S_0) are trivial, we can conclude that the final views (R_n, S_n) also have a trivial distribution.

The next lemma implies that for any δ -private protocol, if the final views have a trivial distribution, then the outputs must satisfy $C(\hat{X}; \hat{Y}) - I(\hat{X}; \hat{Y}) \leq \delta$.

Lemma 7. *Let $C(R; S) = I(R; S)$. If (ϕ, ψ) are deterministic functions such that $I(R; \psi(S)|\phi(R)) + I(S; \phi(R)|\psi(S)) \leq \delta$ then $C(\phi(R); \psi(S)) - I(\phi(R); \psi(S)) \leq \delta$.*

Proof: Let $W_{R,S}$ be the common part of (R, S) . Since ϕ and ψ are deterministic functions, it follows that $\phi(R) \leftrightarrow W_{R,S} \leftrightarrow \psi(S)$ is a Markov chain. Using the property that $W_{R,S}$ is a function of R ,

$$\begin{aligned} I(W_{R,S}; \psi(S)|\phi(R)) &= H(\psi(S)|\phi(R)) - H(\psi(S)|\phi(R), W_{R,S}) \\ &\leq H(\psi(S)|\phi(R)) - H(\psi(S)|\phi(R), R) \\ &= I(R; \psi(S)|\phi(R)). \end{aligned}$$

Similarly, one can show that

$$I(W_{R,S}; \phi(R)|\psi(S)) \leq I(S; \phi(R)|\psi(S)).$$

Thus, $I(W_{R,S}; \psi(S)|\phi(R)) + I(W_{R,S}; \phi(R)|\psi(S)) \leq \delta$, and hence, $C(\phi(R); \psi(S)) - I(\phi(R); \psi(S)) \leq \delta$ by Lemma 3. ■

Thus, if $P_{X,Y}$ can be securely sampled given a set of primitives that do not satisfy the completeness conditions, then for any $\epsilon, \delta > 0$ there exists $P_{\hat{X},\hat{Y}}$ such that $d(P_{\hat{X},\hat{Y}}, P_{X,Y}) \leq \epsilon$ and $C(\hat{X}; \hat{Y}) - I(\hat{X}; \hat{Y}) \leq \delta$. Finally, due to the continuity of Wyner common information (see Lemma 4) and entropy, it follows that $P_{X,Y}$ must be trivial.

B. Achievability Sketch

Due to space restrictions and since the essential techniques are well-known in the literature, we will only sketch the overall scheme for securely sampling any distribution given a primitive satisfying the completeness conditions. Also, we aim only to describe a general but straight-forward approach to show feasibility. Of course, more complex approaches or specialized methods exploiting the structure of particular problem instances may yield more efficient schemes. The overall achievability argument follows these high-level steps:

- 1) Given a primitive satisfying the completeness conditions, we can construct a protocol which can simulate a source primitive $P_{U,V}$ that has a non-trivial distribution.
- 2) The simulated source primitive with a non-trivial distribution can be converted into a binary erasure source via the methods of [9].
- 3) Continuing with the methods of [9], the binary erasure source can be used to perform oblivious transfers.
- 4) Using the methods of [5], general secure computation and hence any secure sampling can be performed via the oblivious transfers.

We further explain these steps below.

Step 1) Let $(R, S) \sim P_{R,S}$ be the random variables and (f, g) be the functions such that the primitive satisfies the completeness conditions. To simulate a source primitive (one with no inputs) with a non-trivial distribution, the parties perform the following:

- Alice generates $R \sim P_R$ independently from her and Bob's current views.
- Alice sends the common part $W_{R,S}$ to Bob via error-free communication.
- Bob generates $S \sim P_{S|W_{R,S}}$ that is conditionally independent from his and Alice's current views given $W_{R,S}$. Note that, since $R \leftrightarrow W_{R,S} \leftrightarrow S$ is a Markov chain, this generation procedure results in $(R, S) \sim P_{R,S}$.
- Alice and Bob interact via the primitive using inputs $A = f(R)$ and $B = g(S)$, and receiving outputs U and V , respectively.

This procedure results in Alice and Bob respectively holding (R, U) and (S, V) that have the non-trivial distribution $P_{(R,U),(S,V)}$ and are independent from their views prior to executing this procedure. Thus, any protocol that requires a source primitive with a non-trivial distribution can equivalently substitute the primitive $P_{U,V|A,B}$ by using this technique. Repeating this procedure generates an iid sequence of sample pairs from the non-trivial distribution $P_{(R,U),(S,V)}$.

Step 2) The methods of [9] require a source primitive $P_{U,V}$ with $H(\tilde{U}|\tilde{V}) > 0$ where (\tilde{U}, \tilde{V}) are the random variables (U, V) with redundancies removed. However, by Lemma 2, this is equivalent to requiring a source primitive with a non-trivial distribution. Due to the properties of distributions with $H(\tilde{U}|\tilde{V}) > 0$, sample pairs from this non-trivial source can be selectively discarded, leaving behind sample pairs that essentially have a binary erasure source distribution, where Alice's sample is a uniform bit and Bob's sample is either equal to Alice's or an erasure symbol (see [9] for details).

Step 3) Using these binary erasure source sample pairs, one can perform oblivious transfer, that is, to essentially simulate the primitive $P_{U,V|A,B}$ where $A := (A_0, A_1)$, $A_0, A_1, B \in \{0, 1\}$, and $(U, V) := (0, A_B)$ (see [9]). Bob first chooses two sample pairs of the binary erasure source for which there is exactly one erasure, and then instructs Alice to respectively exclusive-or her two input bits (A_0, A_1) with the two corresponding bits she has from her half of the erasure source such that the non-erased bit is aligned with the input

that Bob wants (according to B). By sending the result to Bob over the error-free channel, he can recover A_B , while Alice's other bit is masked due to the erasure.

Step 4) Using the methods of [5], the ability to perform oblivious transfers can be leveraged to compute any secure computation, and hence perform any secure sampling. For approximating $P_{X,Y}$ within any variational distance $\epsilon > 0$, the outputs (\hat{X}, \hat{Y}) could be computed from a boolean circuit with a uniformly random sequence of bits as input. Each party first independently generates a uniformly random sequence of bits. Using these as shares of the input sequence, the parties then apply the methods of [5] for securely evaluating the circuit to generate their respective outputs.

Note that evaluating the circuit in the last step requires a fixed number of oblivious transfers; however, the number that can actually be performed depends on the random number of binary erasure sample pairs extracted in the second step. Using a protocol of fixed length (and hence fixed primitive usages), the situation of insufficient erasure samples can be handled as an error event leading to a constant output, and its effect can be made asymptotically small and hence within any ϵ approximation error. This approach also has the benefit of yielding constructions that are perfectly private ($\delta = 0$).

REFERENCES

- [1] Y. Wang and P. Ishwar, "On unconditionally secure multi-party sampling from scratch," in *Proc. IEEE Intl. Symp. on Information Theory*, Saint Petersburg, Russia, Jun. 2011.
- [2] A. Wyner, "The common information of two dependent random variables," *IEEE Transactions on Information Theory*, vol. 21, no. 2, pp. 163–179, Mar. 1975.
- [3] P. Gács and J. Körner, "Common information is far less than mutual information," *Problems of Control and Information Theory*, vol. 2, no. 2, pp. 149–162, 1973.
- [4] M. Ben-Or, S. Goldwasser, and A. Wigderson, "Completeness theorems for non-cryptographic fault-tolerant distributed computation," in *Proc. ACM Symp. on Theory of Computing*, Chicago, IL, 1988, pp. 1–10.
- [5] J. Kilian, "Founding cryptography on oblivious transfer," in *Proc. ACM Symp. on Theory of Computing*, Chicago, IL, 1988, pp. 20–31.
- [6] C. Crépeau and J. Kilian, "Achieving oblivious transfer using weakened security assumptions," in *Proc. IEEE Symp. on the Foundations of Computer Science*, 1988, pp. 42–52.
- [7] C. Crépeau, "Efficient cryptographic protocols based on noisy channels," in *Advances in Cryptology – EUROCRYPT*, ser. Lecture Notes in Computer Science, vol. 1233. Springer-Verlag, 1997, pp. 306–317.
- [8] C. Crépeau, K. Morozov, and S. Wolf, "Efficient unconditional oblivious transfer from almost any noisy channel," in *Proc. Conf. on Security in Communication Networks*, ser. Lecture Notes in Computer Science, vol. 3352. Springer-Verlag, 2004, pp. 47–59.
- [9] A. Nascimento and A. Winter, "On the oblivious transfer capacity of noisy resources," *IEEE Transactions on Information Theory*, vol. 54, no. 6, pp. 2572–2581, Jun. 2008.
- [10] J. Kilian, "More general completeness theorems for secure two-party computation," in *Proc. ACM Symp. on Theory of Computing*, Portland, OR, 2000, pp. 316–324.
- [11] R. Ahlswede and J. Körner, "On common information and related characteristics of correlated information sources," in *Proc. Prague Conf. on Information Theory*, 1974.
- [12] Z. Zhang, "Estimating mutual information via Kolmogorov distance," *IEEE Transactions on Information Theory*, vol. 53, no. 7, pp. 3280–3282, Sep. 2007.
- [13] S. Wolf and J. Wullschleger, "New monotones and lower bounds in unconditional two-party computation," *IEEE Transactions on Information Theory*, vol. 54, no. 6, pp. 2792–2797, Jun. 2008.